TECHNICAL NOTE

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Transverse Vibration of a Simply Supported Beam with Symmetric Overhang of Arbitrary Length

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ABSTRACT: The numerical solution to the frequency equation for the transverse vibration of a simple beam with symmetric overhang is found. The numerical results converge to the analytical solutions for the two limiting cases of a beam with no overhang and a beam with no span and agree with the case in which the supports are at the nodal points of a freely vibrating beam. An approximation to the solution of the frequency equation for beams with small overhang is presented and compared to the numerical solution. This simple yet accurate approximation is most useful to determine a beam's flexural stiffness, *EI,* or modulus of elasticity, *E,* by freely vibrating a simply supported beam.

KEYWORDS: transverse vibration, beam with overhang, flexural stiffness, frequency equation, fundamental frequency

Problem

Over 30 years ago, Pellerin [*1*] investigated the use of transverse vibrations of beams to determine the modulus of elasticity, *E,* of lumber and then predict strength. He examined free vibration of two systems. One was a beam freely supported at two nodal points, and the other was a beam simply supported at the ends. These two systems have analytical solutions to the equations of motion and can be found in the literature [*2,3*]. In these two cases, the supports are located at distances 0.224 times the length of the beam from the ends (nodal points), and at the ends of the beam. In practice, a beam has some overhang and is never supported at its extreme ends. In this report, the vibration of a beam with an overhang of arbitrary length is investigated numerically, an approximate formula for small overhang is proposed, and the results are compared.

Method of Solution

To determine the natural frequency, *f,* of a simply supported beam with symmetric overhang of arbitrary length, we use the methodology used in Timoshenko [*2*] and Seto [*4*] and for brevity refer the reader to these publications. Also, we assume that the cross-sectional dimensions of the beam are constant and small in comparison to its length thereby ignoring the effects of rotary inertia and shearing deformations. #en a beam vibrates transversely in one of its natural modes, the deflection at any location varies harmonically with time, *t*, as follows:

$$y = X(A \cos 2\pi ft + B \sin 2\pi ft)$$

where X is strictly a function, called a normal function, of distance x along the beam and satisfies a fourth-order ordinary differential equation, $X^{iv} - k^4 X = 0$ [*2*]. The general solution to this differential equation has the form:

$$X(x) = C_1 \cos kx + C_2 \sin kx + C_3 \cosh kx + C_4 \sinh kx$$

and for transverse vibration of beams:

$$k^4 = \frac{(2\pi f)^2}{\left(\frac{\rho A}{EI}\right)} \quad (1)$$

where

f = beam natural frequency,
E = beam modulus of elasticity,
I = beam moment of inertia,
ρ = beam mass density, and
A = beam cross-sectional area

The constants C_1 to C_4 must be determined from the boundary conditions at the ends of the beam. Solving for these constants leads to the frequency equation specific for the boundary conditions under consideration.

For our simply supported beam with symmetric overhang we divide the continuous beam into three sections with three distinct coordinate systems and origins. Refer to Fig. 1 for beam geometry. Let X_1, X_2, and X_3 be the normal functions of the beam sections [*4*]. The general solution for the normal functions can be expressed as

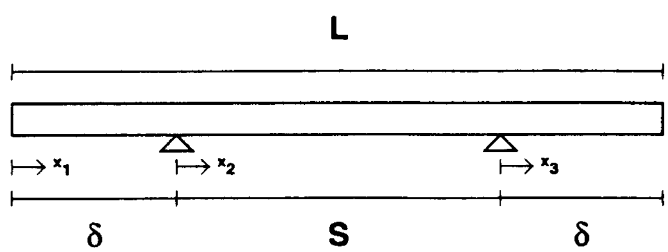

FIG. 1—*Geometry of simply supported beam with symmetric overhang.*

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$$X_1 = A_1 \cos kx_1 + B_1 \cosh kx_1 + C_1 \sin kx_1 + D_1 \sinh kx_1 \quad 0 \le x_1 \le \delta$$
$$X_2 = A_2 \cos kx_2 + B_2 \cosh kx_2 + C_2 \sin kx_2 + D_2 \sinh kx_2 \quad 0 \le x_2 \le \delta$$
$$X_3 = A_3 \cos kx_3 + B_3 \cosh kx_3 + C_3 \sin kx_3 + D_3 \sinh kx_3 \quad 0 \le x_3 \le \delta$$

Along with these three normal functions, we have to satisfy a number of boundary conditions. At the ends of the beam both moment and shear have to be zero. At the supports deflection is zero, slope and moment are continuous. These 12 boundary conditions are expressed mathematically as follows:

$$\text{at } x_1 = 0 \quad d^2X_1/dx_1^2 = 0$$
$$d^3X_1/dx_1^3 = 0$$
$$\text{at } x_1 = \delta,\ x_2 = 0 \quad X_1 = 0$$
$$X_2 = 0$$
$$dX_1/dx_1 - dX_2/dx_2 = 0$$
$$d^2X_1\,dx_1^2 - d^2X_2/dx_2^2 = 0$$
$$\text{at } x_2 = S,\ x_3 = 0 \quad X_2 = 0$$
$$X_3 = 0$$
$$dX_2/dx_2 - dX_3/dx_3 = 0$$
$$d^2X_2/dx_2^2 - d^2X_3/dx_3^2 = 0$$
$$\text{at } x_3 = \delta \quad d^2X_3/dx_3^2 = 0$$
$$d^3X_3/dx_3^3 = 0$$

where δ is beam overhang with $0 < \delta$, S is beam span with $0 < S$, and L is beam length with $S < L$. If we define $\alpha = S/L$ as the ratio of span to length, then the overhang can be expressed as $\delta = L(1 - \alpha)/2$.

From the boundary conditions and the normal functions we can construct a 12 by 12 matrix of the coefficients of the 12 constants. The elements of the matrix consist of the trigonometric and hyperbolic terms of the normal functions. At x_1, x_2, $x_3 = 0$ the arguments of their respective terms are zero. At x_1, $x_3 = \delta$ the arguments of their respective terms are $k\delta$, that is $kL(1 - \alpha)/2$. At $x_2 = S$ the arguments of its terms are kS, that is $kL\alpha$. Therefore all the arguments are either zero or $kL(1 - \alpha)/2$ or $kL\alpha$. If a (span to length ratio) is fixed, then the arguments are a function of kL only. This set of 12 homogeneous equations will have nontrivial solutions only if the determinant of the coefficients vanishes. Expansion of the 12 by 12 determinant is the frequency equation for a simply supported beam with symmetric overhang. Roots of the frequency equation, numerical values of kL forcing the determinant to vanish, correspond to the natural frequencies. We are interested in the first nonzero root, the kL value that corresponds to the natural fundamental frequency. Thus, the minimum nonzero kL value that makes the determinant zero will be used to calculate the fundamental frequency (specific for the overhang corresponding to the chosen a). We rewrite Eq 1 as:

$$f = \frac{k^2}{2\pi}\sqrt{\frac{EI}{\rho A}} = \frac{(kL)^2}{2\pi}\sqrt{\frac{EI}{L^4\rho A}}$$

$$f^2 = \left(\frac{(kL)^2}{2\pi}\right)^2 \frac{EI}{L^4\rho A}$$

We define K_1 as the transformed fundamental root of the frequency equation:

$$K_1 = \left(\frac{(kL)^2}{2\pi}\right)^2$$

$$f^2 = K_1 \frac{EI}{L^4\rho A} = K_1 \frac{EI\,gL}{L^4\,W} \qquad (2)$$

where

g = acceleration of gravity,
W = total beam weight, and
W/gL = beam mass per unit length.

Solution Steps

1. Select a value for α (ratio of span to length S/L) with $1 > \alpha > 0$. (For $\alpha = 1$ or $\alpha = 0$ either the overhangs or the span vanishes and the problem as setup in this report is not valid. These cases have only one normal equation and four boundary conditions.)
2. Choose a value for kL.
3. Find the determinant of the 12 by 12 matrix by
   a. using Gaussian elimination with row pivoting to reduce the matrix to a triangular matrix, and
   b. multiplying the diagonal terms to calculate the determinant.
4. Check the determinant against a very small number and iterate Steps 2, 3, and 4 until the determinant is close enough to zero.
5. Calculate $K_1 = [(kL)^2/(2\pi)]^2$, this K_1 is specific for the α being investigated.
6. Loop Steps 1 to 6 covering S/L, α, from 0.999 to 0.001.

Results

The lower solid curve in Fig. 2 is K_1 computed as described as a function of S/L. As S/L approaches 1, K_1 numerically converges to 2.467 which agrees with the analytical solution [2] of a simply supported beam with no overhang ($[(\pi)^2/(2\pi)]^2$). At $S/L = 0.552$ ($\delta = 0.224\,L$), K_1 is 12.679 which agrees with the analytical solution [2] of a free-free beam with nodal (no deflection) points at 0.224 L and 0.776 L ($[(4.730)^2/(2\pi)]^2$). As S/L approaches 0, K_1 numerically converges to 5.009 which agrees with the adjusted analytical solution [2] of two back-to-back (fixed-end to fixed-end) cantilever beams ($[(1.875 \times 2)^2/(2\pi)]^2$).

If, for small values of δ, we ignore the overhang while still keeping the same beam mass per unit length, we would substitute S^4 for L^4 and use 2.467. The approximation for K_1 is then

$$K_1 \approx 2.467 \frac{L^4}{S^4}$$

and is plotted as the upper dashed curve in Fig. 2. At $S/L = 0.85$ the ratio of the approximation to numerical solution is 1.009, while at $S/L = 0.80$ the ratio is 1.026. Substituting this approximation into Eq 2 results in a simple approximation of the solution to the frequency equation for simply supported beams with symmetric overhang:

$$f^2 \approx 2.467 \frac{EI\,gL}{S^4 W}$$

Conclusions

The numerical solution of the frequency equation of the free transverse vibration of a simply supported beam with symmetric

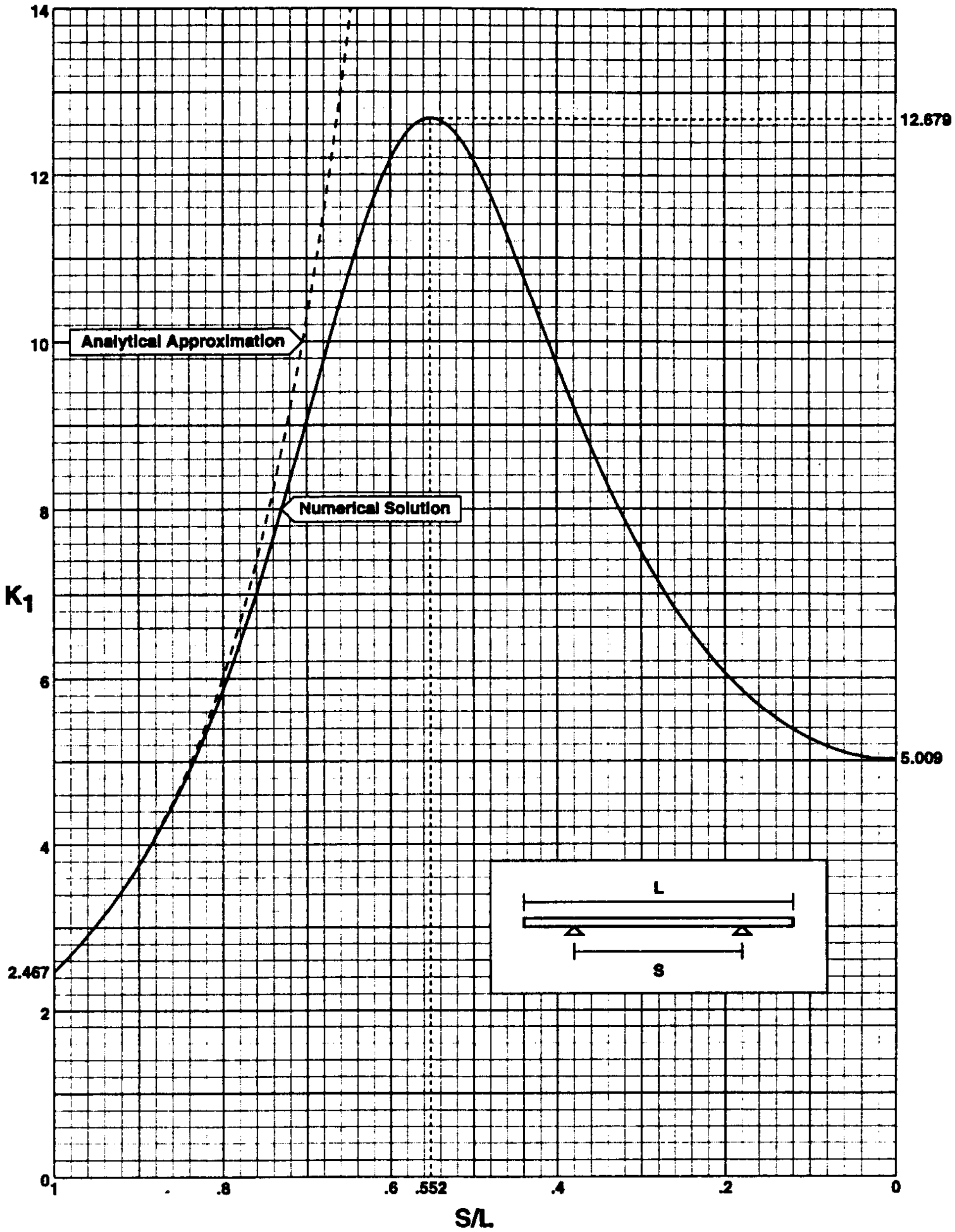

FIG. 2—*Transformed fundamental root of the frequency equation for a simply supported beam with symmetric* ***overhang.***

overhang of arbitrary length is presented. A simple analytical approximation to the numerical solution for the case of small overhang is shown to be quite good for $1 \geq S/L \geq 0.85$ and reasonable for $0.85 \geq S/L \geq 0.80$. The approximation, valid for a simply-supported vibrating beam with small overhang, can be used to compute a beam's flexural stiffness *EI* from measured frequency *f*, measured geometry, *S*, *L*, and measured weight *W* and would result in a conservative estimate of *EI*. The beam's modulus of elasticity *E* can be computed if I is known.

References

[1] Pellerin, R. F., "A Vibrational Approach to Nondestructive Testing of Structural Lumber," *Forest Products Journal*, Vol. 15, No. 3, 1965, pp. 93–101.

[2] Timoshenko, S., Young, D. H., and Weaver, Jr., W., *Vibration Problems in Engineering*, Fourth Edition, John Wiley & Sons, New York, 1974.

[3] Young, W. C., *Roark's Formulas for Stress and Strain*, Sixth Edition, McGraw-Hill Book Company, New York, 1989.

[4] Seto, W. W., *Schaum's Outline Series, Theory and Problems of Mechanical Vibrations*, McGraw-Hill Book Company, New York, 1964.

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